Honors Precalculus One-Sided Limits & Continuity Section 14.3   NOTE TEMPLATE

One-Sided Limits

Limit from the right means that x approaches c from values greater than c.

$$\lim_{x \to c^+} f(x) = L$$

Limit from the left means that x approaches c from values less than c.

$$\lim_{x \to c^-} f(x) = L$$

THEOREM – The Existence of a Limit

Let f be a function and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L if and only if

$$\lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

THIS Theorem means that for a Limit to exist the left-sided and right-sided limits must be equal.

Definition of Continuity

Continuity at a point: A function f is continuous at c, if the following three conditions are met:

i. $f(c)$ is defined

ii. $\lim_{x \to c} f(x)$ exists - (this means $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$)

iii. $\lim_{x \to c} f(x) = f(c)$

Continuity on an Open Interval: a function is continuous on an open interval (a,b) if it is continuous at each point in the interval. A function that is continuous on the entire real line is everywhere continuous.

There are several types of discontinuity.

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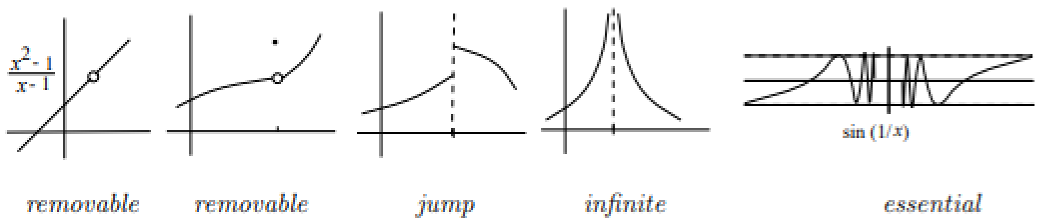
We say a function is **continuous on an interval** $[a, b]$ if it is defined on that interval and continuous at every point of that interval. (At the endpoints, we only use the approrpiate one-sided limit in applying the definition (2).)

A **point of discontinuity** is always understood to be isolated,

Thus, if a is a point of discontinuity, something about the limit statement in (2) must fail to be true.

Types of Discontinuity



In a **removable** discontinuity, $\lim_{x\to a} f(x)$ exists, but $\lim_{x\to a} f(x) \neq f(a)$. This may be because $f(a)$ is undefined, or because $f(a)$ has the "wrong" value. The discontinuity can be removed by changing the definition of $f(x)$ at a so that its new value there is $\lim_{x\to a} f(x)$. In the left-most picture, $\dfrac{x^2-1}{x-1}$ is undefined when $x = 1$, but if the definition of the function is completed by setting $f(1) = 2$, it becomes continuous — the hole in its graph is "filled in".

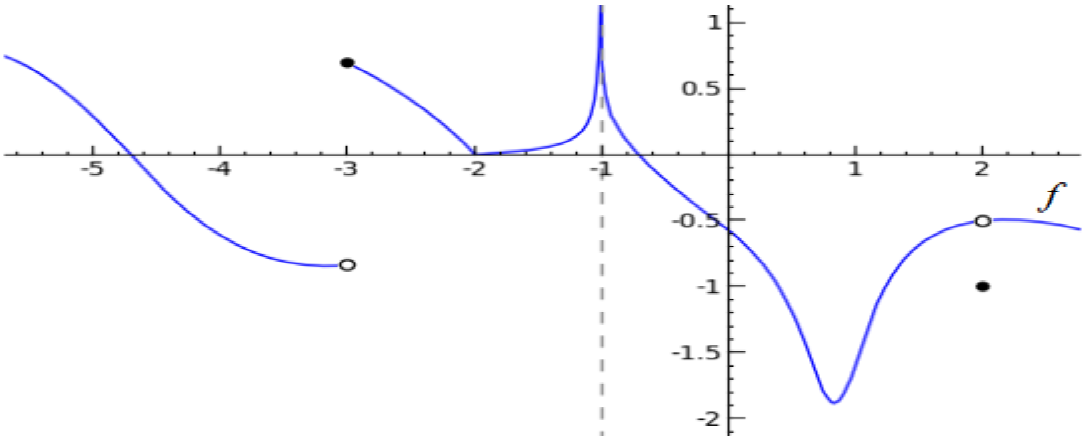
In a **jump** discontinuity (Example 2), the right- and left-hand limits both exist, but are not equal. Thus, $\lim_{x\to a} f(x)$ does not exist, according to (1). The *size* of the jump is the difference between the right- and left-hand limits (it is 2 in Example 2, for instance). Though jump discontinuities are not common in functions given by simple formulas, they occur frequently in engineering — for example, the square waves in electrical engineering, or the sudden discharge of a capacitor.

In an **infinite** discontinuity (Examples 3 and 4), the one-sided limits exist (perhaps as ∞ or $-\infty$), and at least one of them is $\pm\infty$.

An **essential** discontinuity is one which isn't of the three previous types — at least one of the one-sided limits doesn't exist (not even as $\pm\infty$). Though $\sin(1/x)$ is a standard simple example of a function with an essential discontinuity at 0, in applications they arise rarely, presumably because Mother Nature has no use for them.

EXAMPLE # 1:

Given the graph of the piecewise defined function y = f(x) Complete the following:



$\lim_{x \to -3^-} f(x) =$ $\lim_{x \to -3^+} f(x) =$ $\lim_{x \to -3} f(x) =$ Is f(x) continuous at $x = -3$?

$\lim_{x \to -1^-} f(x) =$ $\lim_{x \to -1^+} f(x) =$ $\lim_{x \to -1} f(x) =$ which type of discontinuity is at $x = -1$?

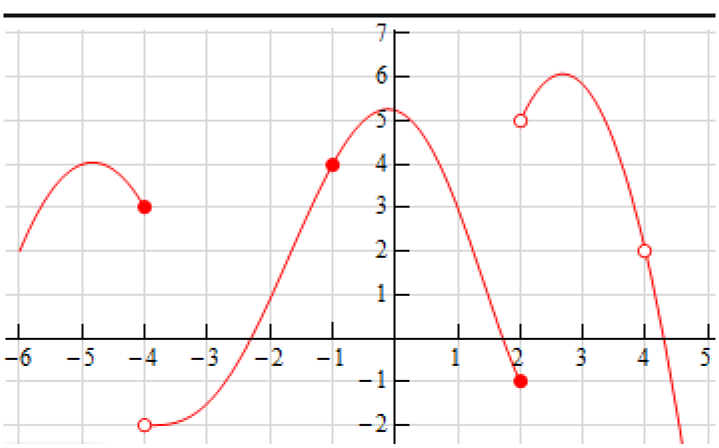
which condition of the definition of continuity fails at $x = -1$?

$\lim_{x \to 0^-} f(x) =$ $\lim_{x \to 0^+} f(x) =$ $\lim_{x \to 0} f(x) =$

$\lim_{x \to 2^-} f(x) =$ $\lim_{x \to 2^+} f(x) =$ $\lim_{x \to 2} f(x) =$

$f(x)$ is discontinuous at $x = 2$. which condition of the definition fo continuity fails and which type of discontinuity is at $x = 2$?

EXAMPLE 2: Use the definition of continuity to determine if y = f(x) is continuous at x = -4, -1, 2, 4



Determine whether f is continuous at c. (This requires you to analyze using the definition of continuity).

$$g(x) = \begin{cases} 8x - 3, & x \le 1 \\ 4x^2 + 5, & x > 1 \end{cases}$$

$$h(x) = \begin{cases} x^2 - 4x + 8, & x \le 3 \\ 2x - 1, & x > 3 \end{cases}$$

$$f(x) = \begin{cases} x^2 - 4, & x \le 0 \\ 2x - 4, & 0 < x < 4 \\ -x^2, & x \ge 4 \end{cases}$$