BC Calculus   Name: ______________________________

10-#1 More Taylor Series Practice

1.

The Maclaurin series for $\ln(1+x)$ is given by

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots.$$

On its interval of convergence, this series converges to $\ln(1+x)$. Let f be the function defined by $f(x) = x\ln\left(1+\frac{x}{3}\right)$.

(a) Write the first four nonzero terms and the general term of the Maclaurin series for f.

(b) Determine the interval of convergence of the Maclaurin series for f. Show the work that leads to your answer.

(c) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Use the alternating series error bound to find an upper bound for $|P_4(2) - f(2)|$.

2.

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$
0	4	5	-1	$-\frac{15}{2}$	23
1	8	3	-2	$\frac{3}{2}$	$\frac{2}{5}$

6. Let f be a function having derivatives of all orders for all real numbers. Selected values of f and its first four derivatives are shown in the table above.

(a) Write the second-degree Taylor polynomial for f about $x = 0$ and use it to approximate $f(0.2)$.

(b) Let g be a function such that $g(x) = f\left(x^3\right)$. Write the fifth-degree Taylor polynomial for g', the derivative of g, about $x = 0$.

(c) Write the third-degree Taylor polynomial for f about $x = 1$.

(d) It is known that $\left|f^{(4)}(x)\right| \le 300$ for $0 \le x \le 1.125$. The third-degree Taylor polynomial for f about $x = 1$, found in part (c), is used to approximate $f(1.1)$. Use the Lagrange error bound along with the information about $f^{(4)}(x)$ to find an upper bound on the error of the approximation.

3.	Let f be the function defined by $f(x) = \dfrac{1}{x^2+9}$. (a) Evaluate the improper integral $\int_3^\infty f(x)\,dx$, or show that the integral diverges. (b) Determine whether the series $\sum_{n=3}^{\infty} f(n)$ converges or diverges. State the conditions of the test used for determining convergence or divergence. (c) Determine whether the series $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{\left(e^n \cdot f(n)\right)} = \sum_{n=1}^{\infty} \dfrac{(-1)^n\left(n^2+9\right)}{e^n}$ converges absolutely, converges conditionally, or diverges.
4.	What is the sum of the series $\frac{\pi}{e} - \frac{\pi}{e^2} + \frac{\pi}{e^3} - \frac{\pi}{e^4} + \cdots + (-1)^{n+1}\frac{\pi}{e^n} + \cdots$? (A) $\dfrac{\pi}{e+\pi}$ (B) $\dfrac{\pi}{e+1}$ (C) $\dfrac{\pi}{e-1}$ (D) The series diverges.
5.	The series $1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots + (-1)^n\frac{x^{2n}}{n!} + \cdots$ converges to which of the following? (A) $\cos\left(x^2\right) + \sin\left(x^2\right)$ (B) $1 - x\sin x$ (C) $\cos x$ (D) e^{-x^2}
6.	Let f be a function such that $f'(x) = \sin\left(x^2\right)$ and $f(0) = 0$. What are the first three nonzero terms of the Maclaurin series for f ? (A) $x - \frac{x^5}{10} + \frac{x^9}{216}$ (B) $2x - x^5 + \frac{x^9}{12}$ (C) $\frac{x^3}{3} - \frac{x^7}{21} + \frac{x^{11}}{55}$ (D) $\frac{x^3}{3} - \frac{x^7}{42} + \frac{x^{11}}{1320}$

7. Which of the following statements about convergence of the series $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$ is true?

(A) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$.

(B) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(C) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$ diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$.

(D) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$ diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

8. Suppose $\lim_{n\to\infty} a_n = \infty$ and $a_{n+1} \ge a_n > 0$ for all $n \ge 1$. Which of the following statements must be true?

(A) $\sum_{n=1}^{\infty} \frac{1}{a_n}$ diverges.

(B) $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

(C) $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges.

(D) $\sum_{n=1}^{\infty} \frac{(-1)^n}{a_n}$ converges.

9. If the power series $\sum_{n=0}^{\infty} a_n(x-4)^n$ converges at $x = 7$ and diverges at $x = 9$, which of the following must be true?

I. The series converges at $x = 1$.

II. The series converges at $x = 2$.

III. The series diverges at $x = -1$.

(A) I only

(B) II only

(C) I and II only

(D) II and III only

10. If the infinite series $S = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n}$ is approximated by $P_k = \sum_{n=1}^{k} (-1)^{n+1} \frac{2}{n}$, what is the least value of k for which the alternating series error bound guarantees that $|S - P_k| < \frac{3}{100}$?

(A) 64 (B) 66 (C) 68 (D) 70

11. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ converges to S. Based on the alternating series error bound, what is the least number of terms in the series that must be summed to guarantee a partial sum that is within 0.03 of S?

(A) 34 (B) 333 (C) 1111 (D) 9999

12.	Which of the following is a power series expansion of $\frac{e^x + e^{-x}}{2}$? (A) $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots$ (B) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$ (C) $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots$ (D) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$
13.	Which of the following statements about the series $\sum_{n=1}^{\infty} \frac{1}{2^n - n}$ is true? (A) The series diverges by the nth term test. (B) The series diverges by limit comparison to the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. (C) The series converges by the nth term test. (D) The series converges by limit comparison to the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

(a) The first four nonzero terms are $\frac{x^2}{3} - \frac{x^3}{2 \cdot 3^2} + \frac{x^4}{3 \cdot 3^3} - \frac{x^5}{4 \cdot 3^4}$.

The general term is $(-1)^{n+1} \frac{x^{n+1}}{n \cdot 3^n}$.

2 : { 1 : first four terms
1 : general term

(b) $$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+2} x^{n+2}}{(n+1)(3^{n+1})}}{\frac{(-1)^{n+1} x^{n+1}}{n \cdot 3^n}} \right| = \lim_{n\to\infty} \left| \frac{-x}{3} \cdot \frac{n}{(n+1)} \right| = \left| \frac{x}{3} \right|$$

$\left| \frac{x}{3} \right| < 1$ for $|x| < 3$

Therefore, the radius of convergence of the Maclaurin series for f is 3.

— OR —

The radius of convergence of the Maclaurin series for $\ln(1+x)$ is 1, so the series for $f(x) = x \ln\left(1 + \frac{x}{3}\right)$ converges absolutely for $\left| \frac{x}{3} \right| < 1$.

$\left| \frac{x}{3} \right| < 1 \Rightarrow |x| < 3$

Therefore, the radius of convergence of the Maclaurin series for f is 3.

When $x = -3$, the series is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-3)^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{3}{n}$, which diverges by comparison to the harmonic series.

When $x = 3$, the series is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{n}$, which converges by the alternating series test.

The interval of convergence of the Maclaurin series for f is $-3 < x \le 3$.

5 : { 1 : sets up ratio
1 : computes limit of ratio
1 : radius of convergence
1 : considers both endpoints
1 : analysis and interval of convergence

— OR —

5 : { 1 : radius for $\ln(1+x)$ series
1 : substitutes $\frac{x}{3}$
1 : radius of convergence
1 : considers both endpoints
1 : analysis and interval of convergence

(c) By the alternating series error bound, an upper bound for $|P_4(2) - f(2)|$ is the magnitude of the next term of the alternating series.

$$|P_4(2) - f(2)| < \left| -\frac{2^5}{4 \cdot 3^4} \right| = \frac{8}{81}$$

2 : { 1 : uses fifth-degree term as error bound
1 : answer

(a) The second-degree Taylor polynomial for f about $x = 0$ is $4 + 5x - \frac{1}{2}x^2$.

$f(0.2) \approx 4 + 5(0.2) - \frac{1}{2}(0.04) = 4.98$

3 :
- 2 : second-degree Taylor polynomial
- 1 : approximation

(b) The fifth-degree Taylor polynomial for g' about $x = 0$ is
$\frac{d}{dx}\left(4 + 5\left(x^3\right) - \frac{1}{2}\left(x^3\right)^2\right) = 15x^2 - 3x^5$.

2 :
- 1 : substitution
- 1 : answer

(c) The third-degree Taylor polynomial for f about $x = 1$ is

$$8 + 3(x-1) - \frac{2}{2}(x-1)^2 + \frac{3/2}{3!}(x-1)^3$$
$$= 8 + 3(x-1) - (x-1)^2 + \frac{1}{4}(x-1)^3.$$

2 :
- 1 : two terms
- 1 : remaining terms

(d) $$\frac{\max\limits_{1 \le x \le 1.1}\left|f^{(4)}(x)\right|}{4!}\cdot(1.1-1)^4 \le \frac{300}{4!}\cdot(1.1-1)^4 = \frac{300}{24}\cdot 0.1^4 = \frac{1}{800}$$

An upper bound on the error of the approximation is $\frac{1}{800}$.

2 :
- 1 : form of the error bound
- 1 : answer

(a) $\int_3^{\infty} \frac{1}{x^2+9}\,dx = \lim_{b\to\infty} \int_3^{b} \frac{1}{x^2+9}\,dx = \lim_{b\to\infty} \left(\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\Big\|_3^b \right)$ $= \lim_{b\to\infty} \left(\frac{1}{3}\tan^{-1}\left(\frac{b}{3}\right) - \frac{1}{3}\tan^{-1}(1) \right) = \frac{\pi}{6} - \frac{\pi}{12} = \frac{\pi}{12}$	3 : { 1 : antiderivative 1 : limit expression 1 : answer
(b) The function f is continuous, positive, and decreasing on $[3, \infty)$. By the integral test, since $\int_3^{\infty} f(x)\,dx$ converges, $\sum_{n=3}^{\infty} f(n)$ converges. — OR — $0 < \frac{1}{n^2+9} < \frac{1}{n^2}$ for $n \ge 3$. Since the series $\sum_{n=3}^{\infty} \frac{1}{n^2}$ converges, the series $\sum_{n=3}^{\infty} f(n) = \sum_{n=3}^{\infty} \frac{1}{n^2+9}$ converges by the comparison test.	2 : conclusion with conditions
(c) Consider the series $\sum_{n=1}^{\infty} \frac{1}{\left(e^n \cdot f(n)\right)} = \sum_{n=1}^{\infty} \frac{n^2+9}{e^n}$. $\lim_{n\to\infty} \left\| \frac{\frac{(n+1)^2+9}{e^{n+1}}}{\frac{n^2+9}{e^n}} \right\| = \lim_{n\to\infty} \left\| \frac{(n+1)^2+9}{n^2+9} \cdot \frac{1}{e} \right\| = \frac{1}{e} < 1$ By the ratio test, $\sum_{n=1}^{\infty} \frac{1}{\left(e^n \cdot f(n)\right)}$ converges. Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\left(e^n \cdot f(n)\right)}$ converges absolutely.	4 : { 1 : sets up ratio 1 : computes limit of ratio 1 : conclusion of ratio test 1 : converges absolutely