

## SOLVING SYSTEMS BY SUBSTITUTION

A system of equations has two or more equations with two or more variables. In Section 4.2, students were introduced to solving a system by looking at the intersection point of the graphs. They also learned the algebraic “equal values” solution method. Graphing and the Equal Values method are convenient when both equations are in  $y$ -form.

The **substitution method** is used to change a two-variable system of equations to a one-variable equation. This method is useful when one of the variables is isolated in one of the equations in the system. Two substitutions must be made to find both the  $x$ - and  $y$ -values for a complete solution. See the Math Notes boxes on pages 248, 252, and 258.

The equation  $x + y = 9$  has infinite possibilities for solutions:  $(10, -1)$   $(2, 7)$   $(0, 9)$ ..., but if  $y = 4$  then there is only one possible value for  $x$ . That value is easily seen when we replace (substitute for)  $y$  with 4 in the original equation:  $x + 4 = 9$ , so  $x = 5$  when  $y = 4$ . Substitution and this observation are the basis for the following method to solve systems of equations.

### Example 1

When solving a system of equations, we replace a variable with an algebraic expression.

Solve:

$$\begin{aligned} y &= 2x \\ x + y &= 9 \end{aligned}$$

$$x + y = 9$$



$$x + 2x = 9$$

$$3x = 9$$

$$x = 3$$

$$\begin{aligned} y &= 2(3) && \text{Use } x = 3 \text{ to substitute in either original equation to find} \\ y &= 6 && \text{the } y \text{ value.} \end{aligned}$$

First replace  $y$  with  $2x$  so there is only one variable in the second equation.

Solve as usual.

The solution is  $(3, 6)$  and makes both equations true. When graphing, this is the intersection point of the two lines.

### Example 2

Substitute the variable that is convenient. Look for the variable that is by itself on one side of the equation.

$$\begin{aligned} \text{Solve: } & 4x + y = 8 \\ & x = 5 - y \end{aligned}$$

Since  $x$  is alone, substitute  $5 - y$  in the first equation for  $x$ , then solve as usual.

$$\begin{aligned} 4(5 - y) + y &= 8 \\ 20 - 4y + y &= 8 \\ 20 - 3y &= 8 \\ -3y &= -12 \\ y &= 4 \end{aligned}$$

$$\begin{aligned} x &= 5 - 4 \\ x &= 1 \end{aligned}$$

Then substitute  $y = 4$  in the original equation to find  $x$ .

The solution is  $(1, 4)$ . Check by substituting this point into each equation to confirm that you are correct.

### Example 3

Not all systems have a solution. If the substitution results in an equation that is not true, then there is NO SOLUTION. This graph would have two parallel lines, so there is no point of intersection.

$$\begin{aligned} \text{Solve: } & y = 7 - 3x \\ & 3x + y = 10 \end{aligned}$$

$$\begin{aligned} 3x + (7 - 3x) &= 10 \\ 3x - 3x + 7 &= 10 \\ 7 &\neq 10 \end{aligned}$$

Substitute  $7 - 3x$  in the second equation for  $y$ .

No Solution.

There may also be an INFINITE NUMBER OF SOLUTIONS. This graph would appear as a single line for the two equations.

$$\begin{aligned} \text{Solve: } & y = 4 - 2x \\ & -4x - 2y = -8 \end{aligned}$$

$$\begin{aligned} -4x - 2(4 - 2x) &= -8 \\ -4x - 8 + 4x &= -8 \\ -8 &= -8 \end{aligned}$$

Substitute  $4 - 2x$  in the second equation for  $y$ .

Always true for any value of  $x$  or  $y$ .

**Example 4**

Sometimes you need to solve one of the equations for  $x$  or  $y$  to use this method. Also be careful when a negative sign appears in the equation. Check your final solution to be sure it works in both of the original equations.

$$\begin{array}{l} \text{Solve: } y - 2x = -7 \\ 3x - 4y = 8 \end{array}$$

$$3x - 4(2x - 7) = 8$$

$$3x - 8x + 28 = 8$$

$$-5x + 28 = 8$$

$$-5x = -20$$

$$x = 4$$

Solve the first equation for  $y$  to get  $y = 2x - 7$ , then substitute  $2x - 7$  in the second equation for  $y$ .

Then substitute again to find  $y$ .  $y = 2(4) - 7$   
 $y = 1$

Check:  $1 - 2(4) = -7$  and  $3(4) - 4(1) = 8$ .

The solution is  $(4, 1)$ . This would be the intersection point of the two lines.

**Problems**

$$\begin{array}{l} 1. \quad y = -3x \\ 4x + y = 2 \end{array}$$

$$\begin{array}{l} 2. \quad y = 7x - 5 \\ 2x + y = 13 \end{array}$$

$$\begin{array}{l} 3. \quad x = -5y - 4 \\ x - 4y = 23 \end{array}$$

$$\begin{array}{l} 4. \quad x + y = 10 \\ y = x - 4 \end{array}$$

$$\begin{array}{l} 5. \quad y = 5 - x \\ 4x + 2y = 10 \end{array}$$

$$\begin{array}{l} 6. \quad 3x + 5y = 23 \\ y = x + 3 \end{array}$$

$$\begin{array}{l} 7. \quad y = -x - 2 \\ 2x + 3y = -9 \end{array}$$

$$\begin{array}{l} 8. \quad y = 2x - 3 \\ -2x + y = 1 \end{array}$$

$$\begin{array}{l} 9. \quad x = \frac{1}{2}y + \frac{1}{2} \\ 2x + y = -1 \end{array}$$

$$\begin{array}{l} 10. \quad a = 2b + 4 \\ b - 2a = 16 \end{array}$$

$$\begin{array}{l} 11. \quad y = 3 - 2x \\ 4x + 2y = 6 \end{array}$$

$$\begin{array}{l} 12. \quad y = x + 1 \\ x - y = 1 \end{array}$$

**Answers**

$$1. \quad (2, -6)$$

$$2. \quad (2, 9)$$

$$3. \quad (11, -3)$$

$$4. \quad (7, 3)$$

$$5. \quad (0, 5)$$

$$6. \quad (1, 4)$$

$$7. \quad (3, -5)$$

$$8. \quad \text{No solution}$$

$$9. \quad (0, -1)$$

$$10. \quad (-12, -8)$$

$$11. \quad \text{Infinite solutions}$$

$$12. \quad \text{No solution}$$

**ELIMINATION METHOD**

In previous work with systems of equations, sections 6.1 and 6.2, one of the variables was usually alone on one side of one of the equations. In that situation, you can substitute to rewrite a two-variable equation as a one-variable equation, then solve the resulting equation.

Systems of linear equations in standard form (that is,  $ax + by = c$ ) have both variables on the same side of both equations. To solve this type of system, we can eliminate one of the variables by focusing on the coefficients. (The coefficient is the number in front of the variable.) This method is called the **elimination method** and uses addition or subtraction, sometimes combined with multiplication or division, to rewrite the system as a one variable equation. After solving for one variable, we then substitute to find the other. A complete solution includes both an  $x$  and a  $y$  value. See the Math Notes boxes on pages 264 and 287 as well as those on pages 252 and 258.

**Example 1**

Solve:  $x - y = 2$   
 $2x + y = 1$       Notice that  $y$  and  $-y$  are opposites.

Adding the two equations together will eliminate the  $y$ -term because  $y + -y = 0$ . This is sometimes called the addition method of elimination.

Add vertically.  $x - y = 2$   
 $+2x + y = 1$   


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 $3x + 0 = 3$   
 Solve for  $x$ .  $x = 1$

Substitute the value of  $x$  in either of the original equations to find the  $y$  value.

$2(1) + y = 1$   
 $2 + y = 1$       The point of intersection is  $(1, -1)$ .  
 $y = -1$

## Example 2

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Solve:  $3x + 6y = 24$   
 $3x + y = -1$

Notice that  $3x - 3x = 0$ .

If the coefficients are the same, subtract the second equation to eliminate the variable, then substitute the value of  $y$  into one of the original equations to find  $x$ .

$$\begin{array}{r} 3x + 6y = 24 \\ -(3x + y = -1) \end{array} \rightarrow \begin{array}{r} 3x + 6y = 24 \\ -3x - y = +1 \\ \hline 0 + 5y = 25 \\ y = 5 \end{array} \quad \text{then} \quad \begin{array}{r} 3x + 6(5) = 24 \\ 3x + 30 = 24 \\ 3x = -6 \\ x = -2 \end{array}$$

The solution is  $(-2, 5)$ . It makes both equations true and names the point where the two lines intersect on a graph.

## Example 3

If neither of the pairs of coefficients are the same, multiply one or both of the equations to make one pair of coefficients the same or opposites. Then add or subtract as needed to eliminate a variable term.

Solve:  $x + 3y = 7$   
 $4x - 7y = -10$

Multiplying the first equation by 4 or -4 will allow you to eliminate the  $x$ . Be sure to multiply everything on the equation's left side and right side to maintain equivalence.

One possible process: multiply the first equation by 4.

$$4(x + 3y = 7) \Rightarrow 4x + 12y = 28$$

Use this equation in the system with the original second equation.

$$\begin{array}{r} 4x + 12y = 28 \\ -(4x - 7y = -10) \\ \hline 19y = 38 \\ y = 2 \end{array} \quad \begin{array}{r} 4x - 7(2) = -10 \\ 4x - 14 = -10 \\ 4x = 4 \\ x = 1 \end{array}$$

Subtract and solve for  $y$ , then substitute in either original equation to find  $x$ .

Solution is  $(1, 2)$ .

If you prefer the addition method, multiply the first equation by  $-4$ :

$$-4(x + 3y = 7) \Rightarrow -4x - 12y = -28. \text{ Then add and solve.}$$

**Example 4**

If one multiplication will not eliminate a variable, multiply both equations by different numbers to get coefficients that are the same or opposites.

Solve:  $8x - 7y = 5$   
 $3x - 5y = 9$

Two ways to eliminate  $x$ :

$$\begin{array}{l} 3(8x - 7y = 5) \Rightarrow 24x - 21y = 15 \\ 8(3x - 5y = 9) \Rightarrow 24x - 40y = 72 \end{array} \quad \text{or} \quad \begin{array}{l} 3(8x - 7y = 5) \Rightarrow 24x - 21y = 15 \\ -8(3x - 5y = 9) \Rightarrow -24x + 40y = -72 \end{array}$$

Two ways to eliminate  $y$ :

$$\begin{array}{l} 5(8x - 7y = 5) \Rightarrow 40x - 35y = 25 \\ 7(3x - 5y = 9) \Rightarrow 21x - 35y = 63 \end{array} \quad \text{or} \quad \begin{array}{l} -5(8x - 7y = 5) \Rightarrow -40x + 35y = -25 \\ 7(3x - 5y = 9) \Rightarrow 21x - 35y = 63 \end{array}$$

The examples above are all equivalent systems and have the same solution of  $(-2, -3)$ .

**Example 5**

The special cases of no solution and infinite solutions can also occur. See the Math Notes box on page 258.

Solve:  $4x + 2y = 6$   
 $2x + y = 3$       Dividing the first equation by 2 produces  $2x + y = 3$ .

The two equations are identical, so when graphed there would be one line with *infinite* solutions because the same ordered pairs are true for both equations.

Solve:  $2x + y = 3$   
 $4x + 2y = 8$       Multiplying the first equation by 2 produces  $4x + 2y = 6$ .

There are no numbers that could make both equations true at the same time, so the lines are parallel and there is *no solution*, that is, no point of intersection.

SUMMARY OF METHODS TO SOLVE SYSTEMS

Method	When to Use	Example
Equal values	Both equations in y-form.	$y = x - 2$ $y = -2x + 1$
Substitution	One variable is alone on one side of one equation.	$y = -3x - 1$ $3x + 6y = 24$
Add to eliminate	Equations in standard form with opposite coefficients.	$x + 2y = 21$ $3x - 2y = 7$
Subtract to eliminate	Equations in standard form with same coefficients.	$x + 2y = 3$ $3x + 2y = 7$
Multiply to eliminate	When nothing else works. In this case you could multiply the first equation by 3 and the second equation by 2, then subtract to eliminate $x$ .	$2x - 5y = 3$ $3x + 2y = 7$

Problems

- |                                    |   |                                       |
|------------------------------------|---|---------------------------------------|
| 1. $2x + y = 6$<br>$-2x + y = 2$   | 2. $-4x + 5y = 0$<br>$-6x + 5y = -10$     | 3. $2x - 3y = -9$<br>$x + y = -2$     |
| 4. $y - x = 4$<br>$2y + x = 8$     | 5. $2x - y = 4$<br>$\frac{1}{2}x + y = 1$ | 6. $-4x + 6y = -20$<br>$2x - 3y = 10$ |
| 7. $6x - 2y = -16$<br>$4x + y = 1$ | 8. $6x - y = 4$<br>$6x + 3y = -16$        | 9. $2x - 2y = 5$<br>$2x - 3y = 3$     |
| 10. $y - 2x = 6$<br>$y - 2x = -4$  | 11. $4x - 4y = 14$<br>$2x - 4y = 8$       | 12. $3x + 2y = 12$<br>$5x - 3y = -37$ |

Answers

- |                 |                         |             |
|-----------------|-------------------------|-------------|
| 1. (1, 4)       | 2. (5, 4)               | 3. (-3, 1)  |
| 4. (0, 4)       | 5. (2, 0)               | 6. Infinite |
| 7. (-1, 5)      | 8. $(-\frac{1}{6}, -5)$ | 9. (4.5, 2) |
| 10. No solution | 11. $(3, -\frac{1}{2})$ | 12. (-2, 9) |